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# A CRITIQUE OF SOME MATHEMATICAL MODELS OF MECHANICAL SYSTEMS WITH DIFFERENTIAL CONSTRAINTS†

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Theories (including mathematical models) put forward for a particular phenomenon are unacceptable, if they do not satisfy certain criteria (see, for instance, [1, p. 14; and 2, pp. 190–196]). The objective criterion of correctness demands that the behaviour predicted by the model agrees with the observed data, to the accepted accuracy.‡ There might be several other correct theories, to the given accuracy.§ In practice, preference is given to the theory which the researcher deems to be the simplest.¶

Courant, in concluding his essay [6, p. 27], emphasizes a fundamental difference between the research formulations and goals of the mathematician and of the scientist solving an applied problem. For the mathematician, the only criterion of the applicability of a theory is its logical consistency. The desire for generality is incompatible with the requirement that the theory should be simple. And since the object of mathematical research is the properties of mathematical relations, even in cases where these relations have arisen in mechanics, for instance, they are usually treated as abstract generalizations, and their correctness (in the above sense) is not discussed.

However, if such research claims to have an applied value, it should, of course, satisfy the criteria of correctness and simplicity. In a number of cases, it is found that the mathematical construction cannot be applied in practice.

This assertion is demonstrated here on the models of Lindelöf and Kozlov.

As WE KNOW, allowance can be made for a finite constraint even in the formulation of the Lagrange function. When solving the problem of the rocking of a solid of revolution about a horizontal plane, Lindelöf also applied this method to a differential constraint. To fix our ideas, we shall call this construction the Lindelöf model, although it had been used before and was to be used subsequently. The significance of his publication lies in the fact that it drew the attention of Chaplygin to the problem [7]. Noting that "even when deriving the differential equations, Lindelöf committed a major error, as a result of which his equations proved to be simpler than the true ones", starting from d'Alembert, Chaplygin derives the equations that now bear his name, which are of much greater significance to research than the successful solution of a specific problem. But Chaplygin solves the problem itself without reference to his equations, and in introducing the reactions of the plane uses general theorems of dynamics on momentum and angular momentum. Observing that he had reduced the problem "to quadratures only in the case where it is reducible . . . to a basic second-order linear differential equation",

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‡ "We are bound to require of every fundamental law of our mechanical system, that when applied to approximately correct relations it should always lead to approximately correct results, not to results which are entirely false." [1, p. 21].

§ Feynman asserts ". . . every theoretical physicist who is any good knows six or seven different theoretical representations for exactly the same physics. He knows that they are all equivalent, and that nobody is ever going to be able to decide which one is right at that level . . ." [3, p. 168].

¶ V. V. Novozhilov: "When any new problem arises, a rough model is first constructed. It is then verified by experiment (if not preceded by it) and better models are constructed, for as long as necessary" [4, p. 361]. At the very beginning of his first lecture, Kirchhoff puts forward the requirement of the simplicity of a theory [5, p. 5], thereby emphasizing the importance of this criterion in the natural sciences.

Chaplygin contrasts his result with that of Lindelöf, which "would appear to solve the problem . . . in its entirety", reducing "everything to the execution of a few quadratures". From this fundamental difference in the analytic forms of the representation of the final results, Chaplygin concludes that the "success" of Lindelöf is only "apparent".

However, it would be desirable to see how these differences show up in the motion of bodies predicted by the Chaplygin and Lindelöf models. Information on all its features can be obtained in graphic form by using modern methods of geometrical representation [8, 9]. But the existence of a set of free parameters requires a considerable amount of work classifying and analysing the possible forms of motion and establishing the differences between them in the models to be compared. Nor is experimental verification very easy. Yet there is an object, the motion of which is well-known: a uniform sphere which rolls without slipping on a horizontal plane. Popular games (such as skittles and billiards) rely on the possibility of the player making a definite prediction of how a uniform sphere will move after receiving an initial impulse. This motion can be assumed to be so well known that the choice of mathematical models on the criterion of correctness in respect of this object is taken as justifiable [2, p. 191]. The incorrectness of the Lindelöf model, and thus its unacceptability in mechanics, is established in just this way below.

Analysing the Kozlov model, which attempted to extend Hamilton's principle to mechanical systems with non-integrable differential constraints, leads to the same conclusion [10–16]. Even Hertz had rejected the possibility of doing this: "The application of Hamilton's principle to a material system does not exclude the existence of fixed connections between the chosen coordinates. But at any rate it still requires that these connections be mathematically expressible by finite equations between the coordinates: it does not permit the occurrence of connections which can only be represented by differential equations . . . Now Hamilton's principle cannot be applied to such a case: or, to speak more correctly, the application, which is mathematically possible, leads to results which are physically false" [1, p. 19]. Poincaré came to the same conclusion [17, p. 328]. Using the simplest example of a sphere rolling by inertia without slipping on a horizontal plane, both Hertz and Poincaré dispense with any mathematical calculations, restricting themselves to qualitative judgements of a kinematic character. Suslov turned to dynamic equations in an examination of the motion of a material particle [18, pp. 362–363]; he proves that "Hamilton's principle cannot be applied to systems subject to non-integrable constraints", since the equations it yields differ from the corresponding equations of Newtonian mechanics.

But if correctness and simplicity are not demanded of the theory thus formulated, which is merely required not to be contradictory, then of course it is quite possible to create an imaginary mathematical construction of a system on the basis of Hamilton's principle, introducing certain relations in the role of non-integrable differential constraints. This is what Kozlov does. However, in so doing he presents his construction as "natural" rather than purely mathematical [10, p. 82], asserting that his "*vako*† dynamics, being an internally consistent model, *applicable to the description of any mechanical systems* [italics and emphasis are mine, P. Kh.], is just as 'true' as the traditional non-holonomic mechanics" [12, p. 110], and accordingly he applies it to classical mechanical systems with non-integrable differential constraints: Chaplygin's sled (in one case, by simplifying the object under consideration [10, p. 99], in others, by making it more complicated [12, pp. 106–109; 15, pp. 552–553]), and the Suslov problem [11, pp. 75–76; 15, pp. 553–554]. In these examples, he confines himself to obtaining (mainly approximate) analytic relations, without reducing them to the form needed in mechanics for comparing calculated and observed (or even expected) "motions", even in those cases where this would be easy to do.

If the Kozlov model is "applicable to the description of any mechanical system", it would seem natural to apply it to an object, the motion of which is widely known and often observed, such as a uniform sphere which rolls by inertia without slipping on a horizontal plane. But Kozlov does not do this, possibly because of Hertz's remark: ". . . a sphere moving in accordance with the [Hamilton's] principle would decidedly have the appearance of a living thing . . . whilst a sphere following the laws of nature would give the impression of an inanimate mass spinning steadily . . ." [1, p. 20].

Notwithstanding the conclusion of Hertz, Poincaré and Suslov, Kozlov still proposes to apply a mathematical model of systems with non-integrable differential constraints based on Hamilton's principle, to mechanics. It has therefore become necessary to give a detailed analysis of the examples mentioned above, to show that application of the Kozlov model yields results that are fundamentally different from observations, in any approximations. This means that the Kozlov model is correct, and cannot be applied to mechanics.

With these considerations, we will first look at the problem of a uniform sphere and, for subsequent

† *Vako* is a new word introduced by V. Kozlov and constructed from the first pairs of letters of his name: *Va*lery *Koz*lov (Editor's note).

comparisons, in a few rows give a solution of that problem on the basis of the general theorems of dynamics. The use of the Lindelöf model for a sphere clearly demonstrates its unacceptability. The unacceptability of the Kozlov model becomes obvious immediately from the predicted trajectories of the centre of the sphere. A complete analysis of the examples considered by Kozlov, the problems of Chaplygin and Suslov, leads to the same conclusion.

## 1. CLASSICAL MODELS

It has been established from observation that, over a certain time interval (for as long as the effect of usually small resistances to rolling and spinning remains unnoticeable), the centre of a uniform sphere, which is situated on a horizontal plane and has received an initial impulse, moves in a straight line, and its axis of rotation maintains its direction in space (as judged from the motion of markers on the surface of the sphere). Choosing fixed coordinate axes in accordance with the initial data, at $t = 0$ we can assume

$$\omega_1 = 0, \quad \omega_2 = n_0, \quad \omega_3 = n \tag{1.1}$$

$$v_1 = v_0, \quad v_2 = 0 \tag{1.2}$$

(as usual $\omega_1$, $\omega_2$ and $\omega_3$ are the components of the angular velocity of the sphere in a fixed system of coordinates with a vertical third axis, $v_1$ and $v_2$ are the components of the velocity of the centre of the sphere in the same system of coordinates, $n_0$ is the initial value of the angular velocity of rolling, $n$ is the initial value of the angular velocity of spinning and $v_0$ is the initial velocity of the centre of the sphere).

The conditions of rolling without slipping

$$f_1(\mathbf{v}, \omega) = v_1 - a\omega_2 = 0, \quad f_2(\mathbf{v}, \omega) = v_2 + a\omega_1 = 0 \tag{1.3}$$

($a$ is the radius of the sphere) give a relation between the initial values

$$v_0 = an_0 \tag{1.4}$$

An elementary solution of the problem of the rolling of a sphere can be obtained from the general theorems of dynamics, by introducing the reaction $(R_1, R_2, R_3)$ at the point where the sphere touches the plane

$$mv_1^{\cdot} = R_1, \quad mv_2^{\cdot} = R_2, \quad J\omega_1^{\cdot} = aR_2, \quad J\omega_2^{\cdot} = -aR_1, \quad J\omega_3^{\cdot} = 0$$

($m$ is the mass of the sphere and $J$ is its moment of inertia about a diameter). Eliminating the reaction $(J\omega_1 - mav_2)^{\cdot} = 0$, $(J\omega_2 + mav_1)^{\cdot} = 0$ and using (1.3), we establish that $J_0\omega_1^{\cdot} = 0$, $J_0\omega_2^{\cdot} = 0$, where

$$J_0 = J + ma^2 \tag{1.5}$$

Therefore, the angular velocity of the sphere (together with the velocity of its centre) keep their initial values (1.1) and (1.2). This result agrees with observations to sufficient accuracy and can be taken to be correct (in the sense indicated above).

But this method of solution is not always used, because not everybody allows the introduction of "obscure and metaphysical" concepts such as reactions [19, p. 24]. The classical methods of formulating the equations of motion, which do not use these concepts, are more laborious. The Euler–Lagrange equations require the calculation of three-index symbols and the Appell equations involve the energy of accelerations. Using these equations in the problem of the rolling without slipping of a uniform sphere on a horizontal plane, we obtain the same correct result (see [20, pp. 372–374, 402] for example).

## 2. THE UNACCEPTABILITY OF THE LINDELÖF MODEL

Following Lindelöf, we will write the kinetic energy of the sphere $T = ½[m(v_1^2 + v_2^2) + J(\omega_1^2 + \omega_2^2 + \omega_3^2)$, using the conditions (1.3) and the notation (1.5): $L = ½[J_0(\omega_1^2 + \omega_2^2) + J\omega_3^2]$. We introduce the Euler angles $\theta$, $\varphi$ and $\psi$, and the kinetic equations

$$\begin{aligned} \omega_1 &= \theta^{\cdot}\cos\psi + \varphi^{\cdot}\sin\theta\sin\psi, \quad \omega_2 = \theta^{\cdot}\sin\psi - \varphi^{\cdot}\sin\theta\cos\psi \\ \omega_3 &= \psi^{\cdot} + \varphi^{\cdot}\cos\theta \end{aligned} \tag{2.1}$$

Then

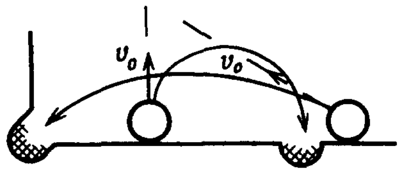


FIG. 1.

$$L = \tfrac{1}{2}[J_0(\theta^{\cdot 2} + \varphi^{\cdot 2}\sin^2\theta) + J(\psi^{\cdot} + \varphi^{\cdot}\cos\theta)^2]$$

The coordinate $\psi$ is cyclic:

$$\omega_3 = \psi^{\cdot} + \varphi^{\cdot}\sin\theta = n \tag{2.2}$$

From (2.1) and (2.2), the equations

$$\left(\frac{\partial L}{\partial \theta^{\cdot}}\right)^{\cdot} - \frac{\partial L}{\partial \theta} = 0, \quad \left(\frac{\partial L}{\partial \varphi^{\cdot}}\right)^{\cdot} = 0$$

can be transformed to

$$\omega_1^{\cdot} + k\omega_2 = 0, \quad \omega_2^{\cdot} - k\omega_1 = 0, \quad k = (J_0 - J)n/J_0 \tag{2.3}$$

Thus, according to Lindelöf, the components $\omega_1$ and $\omega_2$ keep their initial values and the centre of the sphere moves in a straight line only when there is no spinning ($n = 0$). But if $n \neq 0$, it follows from (2.3), the initial data (1.1) and (1.2), and the conditions (1.3) that

$$\omega_1 = -n_0 \sin kt, \quad \omega_2 = n_0 \cos kt$$

$$x_1^{\cdot} = v_1 = an_0 \cos kt, \quad x_2^{\cdot} = v_2 = an_0 \sin kt$$

The trajectory of the centre of the sphere is a circle. According to Lindelöf, it would be possible in billiards, for example, to hit a ball on one side into a hole on the same side without any intermediate collision (Fig. 1). It is common knowledge that motion of this kind is impossible.

The obvious disagreement between the mathematical result and the observed motion means that the Lindelöf model is incorrect and, therefore, unacceptable in mechanics.

## 3. THE UNACCEPTABILITY OF THE KOZLOV MODEL

The mathematical model proposed by Kozlov [10–16] was represented [11] by the system of equations

$$\left(\frac{\partial \mathcal{L}^*}{\partial \omega_i}\right)^{\cdot} + c_{\alpha\beta}^{i}\,\omega_\alpha \frac{\partial \mathcal{L}^*}{\partial \omega_\beta} = X_i(\mathcal{L}^*) \tag{3.1}$$

For a uniform sphere

$$\mathcal{L}^* = \tfrac{1}{2}[J(\omega_1^2 + \omega_2^2 + \omega_3^2) + m(v_1^2 + v_2^2)] - \lambda_1(v_1 - a\omega_2) - \lambda_2(v_2 + a\omega_1) \tag{3.2}$$

To the coordinates $q_1 = \theta$, $q_2 = \varphi$, $q_3 = \psi$, $q_4 = x_1$, $q_5 = x_2$, Kozlov adds, as coordinates, the parameters $\lambda_1$, $\lambda_2$: $q_6 = \lambda_1$, $q_7 = \lambda_2$. Following [11], we put $\omega_4 = v_1$, $\omega_5 = v_2$, $\omega_6 = \lambda_1^{\bullet}$, $\omega_7 = \lambda_2^{\bullet}$, which, together with (2.1) leads to the relations $q_i^{\bullet} = a_{ij}(q)\omega_j$, in which the coefficients can depend only on $\theta$ and $\psi$:

$$a_{ij} = \delta_{ij} \text{ for } i > 3, \quad a_{11} = \cos\psi, \quad a_{12} = \sin\psi, \quad a_{21} = \frac{\sin\psi}{\sin\theta}$$

$$a_{22} = -\frac{\cos\psi}{\sin\theta}, \quad a_{31} = -\frac{\cos\theta}{\sin\theta}\sin\psi, \quad a_{32} = \frac{\cos\theta}{\sin\theta}\cos\psi, \quad a_{33} = 1$$

The other coefficients are zero.

Introducing the operator $X_k = a_{ik}\partial/\partial q_i$ and calculating the commutators $[X_\alpha, X_\beta] = c_{\alpha\beta}^{i}X_i$, we find the non-zero parameter values $c_{32}^1 = -c_{23}^1 = 1$, $c_{13}^2 = -c_{31}^2 = 1$, $x_{21}^3 = -c_{12}^3 = 1$. We then write Eqs (3.1) and the function (3.2) in the form

$$(J\omega_1 - a\lambda_2)^{\cdot} + a\lambda_1\omega_3 = 0 \tag{3.3}$$

$$(J\omega_2 + a\lambda_1)^{\cdot} + a\lambda_2\omega_3 = 0$$

$$J\omega_3^{\cdot} - a(\lambda_1\omega_1 + \lambda_2\omega_2) = 0 \tag{3.4}$$

$$(mv_1 - \lambda_1)^{\cdot} = 0, \quad (mv_2 - \lambda_2)^{\cdot} = 0 \tag{3.5}$$

$$v_1 - a\omega_2 = 0, \quad v_2 + a\omega_1 = 0 \tag{3.6}$$

We introduce the new variables $\kappa$ and $\sigma$, putting

$$\lambda_1 = ma\kappa\cos\sigma, \quad \lambda_2 = ma\kappa\sin\sigma \tag{3.7}$$

We express the constants of integration of Eqs (3.5) in terms of constants $c$ and $\epsilon$: $v_1 = a(\kappa\cos\sigma + c\cos\epsilon)$, $v_2 = a(\kappa\sin\sigma + c\sin\epsilon)$ and, obviously, without loss of generality, we can take $c \geq 0$. From (3.6) it then follows that

$$\omega_1 = -\kappa\sin\sigma - c\sin\epsilon, \quad \omega_2 = \kappa\cos\sigma + c\cos\epsilon \tag{3.8}$$

We substitute these values and those of (3.7) into (3.3) and find that

$$\kappa = \text{const} \tag{3.9}$$

$$\omega_3 = J_0(ma^2)^{-1}\sigma^{\cdot} \tag{3.10}$$

If $\lambda_1 = \lambda_2 = 0$, from (3.3)–(3.5) we obtain the equations of motion of a sphere on a perfectly smooth plane $J\omega_i^{\bullet} = 0$, $(i = 1, 2, 3)$, $mv_j^{\bullet} = 0$ $(j = 1, 2)$, which differ from those of Sec. 1 in the absence of horizontal components $R_1$, $R_2$, of the constraint reactions. But if $\lambda_1^2 + \lambda_2^2 \neq 0$, from (3.7) and (3.9) it follows that the constant $\kappa$ can always be taken as positive: $\kappa > 0$. After allowing for relations (3.1), (3.7), (3.8) we write the last unused equation (3.4) as: $\sigma^{\bullet\bullet} - m^2a^4(JJ_0)^{-1}c\kappa\sin(\sigma - \epsilon) = 0$. We represent the constant arising from the first integration in terms of the arbitrary parameter $k$:

$$\sigma^{\cdot 2} = 4\frac{m^2a^4}{JJ_0}c\kappa\left(k^2 - \cos^2\frac{\sigma - \epsilon}{2}\right)$$

Putting

$$\tau = n_* t, \quad n_* = ma^2\sqrt{c\kappa/JJ_0} \tag{3.11}$$

and, changing to Jacobi elliptic functions [21] we have

$$\cos\frac{\sigma - \epsilon}{2} = k\,\mathrm{sn}(\tau, k), \quad \sin\frac{\sigma - \epsilon}{2} = -\mathrm{dn}(\tau, k)$$

$$\frac{d\sigma}{d\tau} = k\,\mathrm{cn}(\tau, k)$$

and from (3.8), (3.10) and (3.11)

$$\begin{aligned} \omega_1 &= -[c + \kappa - 2\kappa\,\mathrm{dn}^2(\tau, k)]\sin\epsilon + \\ &+ 2\kappa k\,\mathrm{sn}(\tau, k)\,\mathrm{dn}(\tau, k)\cos\epsilon \\ \omega_2 &= [c + \kappa - 2\kappa\,\mathrm{dn}^2(\tau, k)]\cos\epsilon + \\ &+ 2\kappa k\,\mathrm{sn}(\tau, k)\,\mathrm{dn}(\tau, k)\sin\epsilon \\ \omega_3 &= k\sqrt{c\kappa J_0/J}\,\mathrm{cn}(\tau, k) \end{aligned} \tag{3.12}$$

We satisfy the initial conditions (1.1): $(c - \kappa)\sin\epsilon = 0$, $(c - \kappa)\cos\epsilon = n_0$, $k\sqrt{(c\kappa J_0/J)} = n$. And since, in the general case, $n_0 \neq 0$, we have

$$\epsilon = 0, \quad c = n_0 + \kappa, \quad k = n\sqrt{J/[(n_0 + \kappa)\kappa J_0]} \tag{3.13}$$

and thus

$$\begin{aligned} &\omega_1 = 2\kappa k\,\mathrm{sn}(\tau, k)\,\mathrm{dn}(\tau, k), \quad \omega_2 = n_0 + 2\kappa k^2\,\mathrm{sn}^2(\tau, k) \\ &\omega_3 = n\,\mathrm{cn}(\tau, k) \end{aligned} \tag{3.14}$$

The parameter $\kappa$ is still arbitrary.

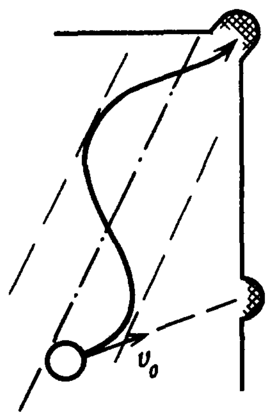


FIG. 2.

We obtain the trajectory of the centre of the sphere by integration from (3.6), using (3.11), (3.13), (3.14) and (1.4)

$$x_1(t) = (v_0 + 2a\kappa)t - 2\frac{a}{n_*}\kappa \int_0^{n_* t} dn^2(\tau, k)\, d\tau$$
$$x_2(t) = 2\frac{nJ}{(n_0 + \kappa)ma}\operatorname{cn}(n_* t, k) \quad (3.15)$$

To be specific, we will restrict ourselves to values $\kappa \in ]0; 1[$ which, from (3.13), we obtain by subjecting $\kappa$ to the condition $\kappa > \sqrt{n_0^4/4 + JJ_0^{-1}n^2} - n_0/2$.

The trajectory (3.14) successively touches the boundaries of the strip

$$-2nJ/[(n_0 + \kappa)ma] \leq x_2 \leq 2nJ/[(n_0 + \kappa)ma]$$

and the quantity

$$x_1^{\cdot} x_2^{\cdot\cdot} - x_2^{\cdot} x_1^{\cdot\cdot} = a^2 k n_* \kappa \operatorname{cn}(n_* t, k)\{\kappa + (n_0 + \kappa)[1 - 2\operatorname{dn}^2(n_* t, k)]\}$$

becomes zero at $x_2 = 0$. Thus, the centre of the sphere should describe an undulating curve in the plane (Fig. 2). But motion of this kind is not observed and, therefore, the Kozlov model, like that of Lindelöf, leads to a mathematical result which is inconsistent with the observed motion, which means that it is unacceptable in mechanics.

Also, it is not single-valued, and this compounds the difficulty. The initial conditions (1.1) do not, as is proper in problems of mechanics, identify a unique solution that is acquired by the sphere under those initial conditions among the general solution (3.12) of Eqs (3.3)–(3.6). That this is inevitable follows from the construction of the model. Considering the problem as a variational one with restrictions (differential constraints), the author introduces a Lagrangian with parameters $\lambda_i$. He assumes these to be "extra parameters" [11, p. 71]. But since these "coordinates" have no mechanical meaning, there are no rational premises for assigning specific initial conditions to them that correspond to the problem. The constants of integration that correspond to these "coordinates" must remain arbitrary, which is the reason why the resulting solution is not unique, even when the true coordinates satisfy the initial data. Thus, the Kozlov model is only a mechanical construction and does not pertain to mechanics.

## 4. CHAPLYGIN'S SLED AND THE SUSLOV PROBLEM

Experiments on the rolling of a body on a surface without slipping are easy to perform. Bodies of the simplest shapes (spheres or discs), with relatively little idealization, are suitable for analytic study as well. At the same time, in the theory of non-holonomic systems, use is made of "imaginary" constructions which, although they are connected with real objects, involve considerable idealization of their properties. These include "Chaplygin's sled" [22, pp. 21–24], and the "non-twisting thread" in the problem of Suslov [18, pp. 593–594]. Yet these are the very examples that are used [10–12, 15].

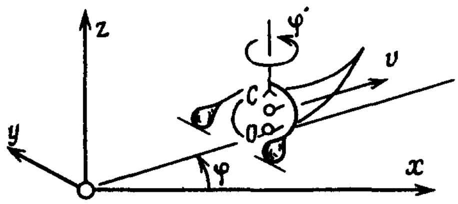


FIG. 3.

We will show that a more detailed analysis of the motion in the Chaplygin and Suslov problems based on the Kozlov theory also yields results which must be regarded as unacceptable in mechanics.

Chaplygin's sled has been called a "skate" in the case where the centre of mass $C$ is above the point $O$ where the runner touches the horizontal plane [10, p. 99] (Fig. 3). It is assumed that there is no friction of slipping or spinning, and the non-holonomic constraint $x^{\cdot}\sin\varphi - y^{\cdot}\cos\varphi = 0$ implies that the velocity vector **v** of the point $C$ remains in the plane of the runner

$$x^{\cdot} = v\cos\varphi, \quad y^{\cdot} = v\sin\varphi \tag{4.1}$$

All the classical methods [20, 22, 23] of solving this problem lead to the same result: if at the initial instant

$$v = v_0 \neq 0, \quad \varphi^{\cdot} = \omega_0 = 0 \tag{4.2}$$

the skate slides uniformly along the straight line $x = v_0 t$, $y \equiv 0$, $\varphi \equiv 0$ (the unimportant arbitrary constraints $x_0$, $y_0$, $\varphi_0$ are eliminated by the choice of coodinate axes). This result is regarded as acceptable in mechanics (it can be construed as the idealized model of the motion of a skater who, after pushing himself off, slides smoothly on the ice).

The Kozlov model gives a different result. It introduces the canonical impulses [10, p. 99]

$$p_x = x^{\cdot} - \lambda\sin\varphi, \quad p_y = y^{\cdot} + \lambda\cos\varphi, \quad p_\varphi = \varphi \tag{4.3}$$

with the extra variable

$$\lambda = p_y\cos\varphi - p_x\sin\varphi \tag{4.4}$$

and the Hamiltonian function

$$H = \tfrac{1}{2}[(p_x\cos\varphi + p_y\sin\varphi)^2 + p_\varphi^2] \tag{4.5}$$

Later [11, p. 73] the variable $l$ is called an "extra coordinate".

From (4.5) we have

$$p_x^{\cdot} = 0, \quad p_y^{\cdot} = 0 \tag{4.6}$$

$$p_\varphi^{\cdot} = (p_x\cos\varphi + p_y\sin\varphi)(p_x\sin\varphi - p_y\cos\varphi) \tag{4.7}$$

We will express the constants that appear in the integration of Eqs (4.6) in terms of the two parameters $c$ and $\epsilon$:

$$p_x = c\sin\epsilon, \quad p_y = c\cos\epsilon$$

and from (4.4), (4.3) and (4.7), using (4.1) and (4.2), we have

$$\lambda = c\cos(\varphi + \epsilon), \quad v = c\sin(\varphi + \epsilon)$$

$$\varphi^{\cdot 2} = c^2[\sin^2\epsilon - \sin^2(\varphi + \epsilon)]$$

from which we obtain [21]

$$\sin(\varphi + \epsilon) = k\,\mathrm{sn}(\tau, k), \quad \cos(\varphi + \epsilon) = \mathrm{dn}(\tau, k),$$

$$k = \sin\epsilon, \quad k' = \cos\epsilon, \quad \tau = K(k) + ct$$

$$K(k) = \int_0^1 \frac{dz}{\sqrt{(1 - z^2)(1 - k^2 z^2)}}$$

and from (4.1)

$$x(\tau) = k[-k'\mathrm{cn}(\tau, k) + k^2 \int_K^\tau \mathrm{sn}^2(z, k)\,dz]$$

$$y(\tau) = k^2[\mathrm{cn}(\tau, k) + k' \int_K^\tau \mathrm{sn}^2(z, k)\,dz]$$

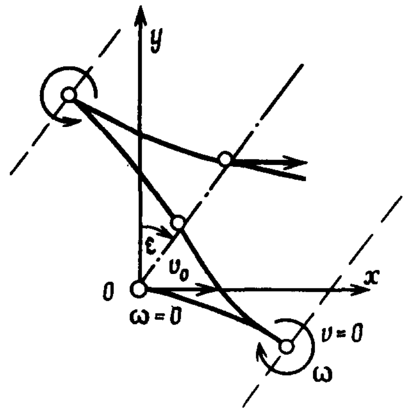


FIG. 4.

These equations define the fanciful track of a skate that has received the initial impulse (4.2) (Fig. 4). It can hardly be regarded as acceptable in mechanics. Yet it is, of course, a consequence of the *a priori* principle on which the Kozlov theory is based.

As in the case of a sphere, the unacceptability of the Kozlov model of the Chaplygin problem is aggravated by the presence in the solution of the arbitrary parameter $\epsilon$, as a result of which the initial conditions (4.2) do not define a unique solution. Nor does the existence in this set of the trajectory $x = v_0 t$, $y \equiv 0$, $\varphi \equiv 0$, obtained for $\epsilon = \pi/2$, save the situation, because there is no basis on which to assign a specific value to $\epsilon$.

Now let us turn to Suslov's problem.

The system of equations

$$\begin{gathered} J_{11}\omega_1^{\cdot} + (J_{13}\omega_1 + J_{23}\omega_2 - \lambda)\,\omega_2 = 0 \\ J_{22}\omega_2^{\cdot} - (J_{13}\omega_1 + J_{23}\omega_2 - \lambda)\,\omega_1 = 0 \\ (J_{13}\omega_1 + J_{23}\omega_2)^{\cdot} + (J_{22} - J_{11})\,\omega_1\omega_2 = \mu \end{gathered} \tag{4.8}$$

with

$$\lambda = 0 \tag{4.9}$$

is the same as that obtained by Suslov, and when

$$\lambda \neq 0, \quad \mu = \lambda^{\cdot} \tag{4.10}$$

these are the equations proposed by Kozlov [11, p. 75].† Using the integral $J_{11}\omega_1^2 + J_{22}\omega_2^2 = h^2$, following Suslov, we introduce the variable $\chi$.

$$\omega_1 = \frac{h}{\sqrt{J_{11}}}\sin\chi, \quad \omega_2 = \frac{h}{\sqrt{J_{22}}}\cos\chi$$

and substitute these expressions into the initial system (4.8):

$$\chi^{\cdot}\sqrt{J_{11}J_{22}} + \left(\frac{J_{13}}{\sqrt{J_{11}}}\sin\chi + \frac{J_{23}}{\sqrt{J_{22}}}\cos\chi\right)h = \lambda \tag{4.11}$$

$$\left(\frac{J_{13}}{\sqrt{J_{11}}}\cos\chi - \frac{J_{23}}{\sqrt{J_{22}}}\sin\chi\right)h\chi^{\cdot} + \frac{J_{22} - J_{11}}{\sqrt{J_{11}J_{22}}}h^2\sin\chi\cos\chi = \mu \tag{4.12}$$

In Suslov's formulation [with the value (4.9)], Eq. (4.11) establishes the dependence of $\chi$ on $t$ in terms of elementary functions:

$$\operatorname{tg}\frac{\chi + \alpha}{2} = \operatorname{tg}\frac{\chi_0 + \alpha}{2}e^{-kt}$$

$$\operatorname{tg}\alpha = \frac{J_{23}}{J_{13}}\sqrt{\frac{J_{11}}{J_{22}}}, \quad k = h\sqrt{\frac{J_{13}^2}{J_{11}^2 J_{22}} + \frac{J_{23}^2}{J_{22}^2 J_{11}}}$$

†The unimportant component $J_{12}$ of the inertial tensor that is retained in [11] has been omitted from Eqs (4.8).

The important point here is that it is the components $J_{13}$ and $J_{23}$ of the inertial tensor characterizing the dynamic disbalance of the body relative to the third axis associated with the constraint that determine all the features of motion of the body. When $J_{13} = J_{23} = 0$, the body rotates uniformly about a fixed axis.

The "imaginary" model of Kozlov is a different matter. Under conditions (4.10), we eliminate $\lambda$ from Eqs (4.11) and (4.12)

$$\chi^{\cdot\cdot} + \frac{J_{11} - J_{22}}{J_{11} J_{22}} h^2 \sin \chi \cos \chi = 0$$

and thus the values of the components $J_{13}$ and $J_{23}$ in (4.8) have no effect on the dependence $\chi(t)$ [or on $\omega_1(t)$ and $\omega_2(t)$]. But this means that in the Kozlov model any disbalance present in the body does not show up in its motion. A result of this kind must be regarded as unacceptable in mechanics.

## 5. "ON REALIZING A CONSTRAINT"

The examples given above thus repudiate the assertion of Kozlov that his construction is applicable to the description of the motion of *any* mechanical system. Further, having made this assertion, Kozlov immediately introduces a restriction that contradicts it: "If the non-integrable relations are realized by viscous friction forces with a high coefficient of viscosity, it is natural to use a non-holonomic model to describe the motion . . . But if constraints arise as the result of a change in the Riemann metric of the system (with the help of added masses, for instance), then from a theoretical point of view, preference must be given to the *vako* model" [12, p. 110]. Thus, Kozlov makes an unexpected demand on mechanics: already knowing the equation of the constraint, introduced whilst declining to use information on the type of interaction between a given body and the body realizing the constraint, he writes out the equations of motion of the body only for the case where that information exists. The phrase that Kozlov often uses, "realizing a constraint", is not explained.

When constructing a mathematical model of this motion in applied mechanics, we are always dealing with a specific system of interacting bodies. Some of the interactions might be characterized by Newtonian forces, the dependence of which on the mutual positions of the bodies and their relative velocities has been established by experiment (observation). In those cases where the system includes bodies which are in contact, the physico-chemical properties of the bodies in the contact regions are very important, but as yet unpredictable. Whilst these interactions might be characterized by means of forces, nevertheless this is only on the basis of empirical dependences, which normally have very narrow ranges of application. The results of many different experiments have been collated, classified and published for reference in the solution of technical problems. In those cases where it is regarded as acceptable to ignore the physical parameters of interaction, the introduction of any hypothesis concerning the mechanism of interaction is totally rejected, dynamic factors are ignored, and only geometrical (kinematic) characteristics of the motion and observed restrictions caused by a body in contact with the given body are considered. "If the first body touches other bodies which restrict its freedom of movement in one way or another, these bodies are called constraints in relation to the first" [24, p. 19]. Definitions of the concept of a constraint in terms of material bodies like this are usually introduced in theoretical systems intended for use in applications. However, the concept of a constraint has also been introduced in a different way, as a *designation* ascribed to a mathematical relation—this is the *a priori* axiomatic introduction of an initial concept that is usual in mathematics: "Let $\gamma$ be an $m$-dimensional surface in $3n$-dimensional configurational space of points $\mathbf{r}_1, \dots, \mathbf{r}_n$ of mass $m_1, \dots, m_n$. Let $q = (q_1, \dots, q_m)$ be any coordinates on $\gamma: \mathbf{r}_i = \mathbf{r}_i(q)$. The system described by the equations

$$\frac{d}{dt}\frac{\partial L}{\partial q^{\cdot}} = \frac{\partial L}{\partial q}, \quad L = \frac{1}{2}\Sigma m_i \mathbf{r}_i^{\cdot 2} + U(q),$$

is called a system of $n$ points, constrained by $3n - m$ ideal holonomic constraints" [25, pp. 66–67]. Kozlov too introduces the term "constraint" as the designation of a mathematical relation (in the more general case, a differential relation) [10, p. 92; 12, p. 102].

If the concept of a constraint arises in the mathematical modelling of the interactions of material bodies, without regard to the dynamic characteristics of interaction, then the question of "realizing the constraint" has no meaning and does not arise in mechanics. But if the mathematical relation, called the constraint, is taken as the primary object, then in the mechanics, which has met this kind of thing in the formulation of the problem, there is a natural tendency to attempt to establish the physical meaning of this equation. The search for such a meaning is called "realizing a constraint".

This phase seems to have appeared for the first time in a discussion of the problem of Chaplygin's sled. Fufayev [26] refers to Caratheodory's conclusion that, instead of the unknown reaction of the plane to the runner, viscous friction forces should be introduced and a "constraint realized" by letting the coefficient of viscosity tend to infinity. The artificiality of this formulation is obvious. Viscous friction characterizes the effect of a fluid medium in which the assumption of anisotropy (especially of a runner that has lost its position) is unnatural. The mathematical construction suggested by Caratheodory was later applied to a system of general form [27].

A fundamentally different interpretation of "realizing a constraint" was proposed by Kozlov. He first used this phrase in connection with the problem of Suslov "on the rotation of a solid with zero projection of angular velocity on a fixed axis in the body", in a proposed discussion of a procedure in which "this constraint is realized with the help of Kirchhoff's problem of the motion of a solid in an infinite ideal fluid" [12, p. 105]. In the Chaplygin problem of a skate, Kozlov suggested "realizing the constraint" by "the motion, in an infinite ideal fluid, of a long weightless elliptical plate with rigidly fixed points of positive mass" [12, p. 109]. It is clear that there is scarcely any physical meaning in this method of "realizing the constraints" of one mechanical object by means of another which has no physical connection with the first.

Considering the classical problem of the motion in an ideal fluid (which extends to infinity and is at rest at infinity) of a body in the shape of an ellipsoid with semi-axes $a = \epsilon$, $b = \epsilon^{-\alpha}$, $c$ ($\epsilon$ and $\alpha$ are positive), Kozlov first assumes $c = 0$ and then passes to the limit as $\epsilon \to 0$.

The very fact that, for the limiting "infinite straight line", under the given conditions at infinity, the problem of hydrodynamics has a trivial solution (an ideal fluid at rest, which "does not see" such an object) which differs from that proposed by Kozlov, indicates that his result is incorrect. The "*vako* mechanics" of Kozlov is not needed in the limiting problem. It is an ordinary Lagrangian system [28, pp. 234–237] which, of course, can also be described by the ordinary Hamilton equations.

Thus, in mechanics there are no objects to which Kozlov's "*vako*" model has to be applied.

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# THE PROBLEM OF REALIZING CONSTRAINTS IN DYNAMICS†

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Problems associated with the limiting transition in the second-order Lagrange equations, when the coefficients of rigidity and viscosity and added masses tend to infinity are considered. Under certain conditions, the solutions of the initial equations approach those of the limiting problem with constraints. For integrable constraints, the limiting equations are identical with the usual equations with constraint multipliers. In the case of non-integrable constraints, the solutions depends closely on the way in which they are realized. The generalized models of the dynamics of systems with non-integrable constraints and the properties of the limiting equations of motion are discussed.

**1.** LET $x_1, \ldots, x_n$ be generalized coordinates of a mechanical system, let $T$ be its kinetic energy and $F_1, \ldots, F_n$ generalized forces. If the system is "free" (that is, the coordinates $x$ and velocities $x^\bullet$ are not subject to a non-trivial relation), then its motions can be described by the Lagrange equations

$$[T] = F \tag{1.1}$$

where $[f]$ is the variational derivative $(\partial f/\partial x^\bullet) - \partial f/\partial x$.

If there is a constraint $\Phi(x^\bullet, x, t) = 0$ (in applications, the function $\Phi$ is linear in $x^\bullet$), then Eqs (1.1) are usually replaced by the more general equations

$$[T] = F + \lambda \partial\Phi/\partial x^\bullet, \quad \Phi = 0, \tag{1.2}$$

where $\lambda$ is an as yet undefined multiplier. Let $\partial\Phi/\partial x^\bullet \neq 0$. Then $\lambda$ can be put in the form of an explicit function of $x^\bullet$, $x$ and $t$ without solving (1.2).

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